

MULTIPLIERS ON HILBERT SPACES OF DIRICHLET SERIES

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ABSTRACT. In this paper, certain classes of Hilbert spaces of Dirichlet series with weighted norms and their corresponding multiplier algebras will be explored. For a sequence $\{w_n\}_{n=n_0}^{\infty}$ of positive numbers, define

$$\mathcal{H}^w = \left\{ \sum_{n=n_0}^{\infty} a_n n^{-s} : \sum_{n=n_0}^{\infty} |a_n|^2 w_n < \infty \right\}.$$

Hedenmalm, Lindqvist and Seip considered the case in which $w_n \equiv 1$ and classified the multiplier algebra of \mathcal{H}^w for this space in [3]. In [4], McCarthy classified the multipliers on \mathcal{H}^w when the weights are given by

$$w_n = \int_0^{\infty} n^{-2\sigma} d\mu(\sigma),$$

where μ is a positive Radon measure with $\{0\}$ in its support and n_0 is the smallest positive integer for which this integral is finite. Similar results will be derived assuming the weights are multiplicative, rather than given by a measure. In particular, upper and lower bounds on the operator norms of the multipliers will be obtained, in terms of their values on certain half planes, on the Hilbert spaces resulting from these weights. Finally, some number theoretic weight sequences will be explored and the multiplier algebras of the corresponding Hilbert spaces determined up to isometric isomorphism, providing examples where the conclusion of McCarthy's result holds, but under alternate hypotheses on the weights.

1. INTRODUCTION

A *Dirichlet series* is a series of the form

$$(1) \quad \sum_{n=1}^{\infty} a_n n^{-s},$$

where the a_n 's and s are complex numbers. Such a series may or may not converge, depending on the a_n 's and the choice for s . For example, if $a_n = n!$ for each n , then the series fails to converge anywhere. If instead a_n is nonzero for only finitely many n , then the series converges everywhere. It turns out that if a Dirichlet series converges for some complex number s_0 , then it must converge for all complex s with real part, $\Re(s)$, strictly larger than s_0 . Given a real number δ , let Ω_{δ} denote

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the open half-plane

$$\{z \in \mathbb{C} : \Re(z) > \delta\}.$$

By the preceding remark, if a Dirichlet series converges at s_0 , then it converges in $\Omega_{\Re(s_0)}$.

We are going to be concerning ourselves with Hilbert spaces of functions representable by Dirichlet series in some open half-plane. Denote by \mathcal{D} the set of those functions whose domain contains some open right half-plane (which right half-plane will generally depend on the function) in which the function is representable by a Dirichlet series. Let \mathcal{D}_δ denote the set of functions that can be represented by a Dirichlet series specifically in the right half-plane Ω_δ .

Let $\{w_n\}_{n=n_0}^\infty$ be a sequence of positive real numbers and suppose that there is a positive real number σ_0 such that

$$(2) \quad \sum_{n=n_0}^\infty w_n^{-1} n^{-2\sigma} < \infty$$

whenever $\sigma > \sigma_0$. Let \mathcal{H}^w denote the Hilbert space

$$(3) \quad \left\{ \sum_{n=n_0}^\infty a_n n^{-s} : \sum_{n=n_0}^\infty |a_n|^2 w_n < \infty \right\},$$

with inner product $\langle \cdot, \cdot \rangle_w$ defined by

$$\left\langle \sum_{n=n_0}^\infty a_n n^{-s}, \sum_{m=n_0}^\infty b_m m^{-s} \right\rangle_w = \sum_{n=n_0}^\infty a_n \overline{b_n} w_n.$$

For simplicity, we will take $n_0 = 1$ unless otherwise stated. Note that (2), along with the Cauchy-Schwarz inequality, implies that $\mathcal{H}^w \subseteq \mathcal{D}_{\sigma_0}$.

A *multiplier* on \mathcal{H}^w is a function φ , with domain containing the half-plane Ω_{σ_0} , with the property that $\varphi f \in \mathcal{H}^w$ for each $f \in \mathcal{H}^w$. Thus, φ determines a mapping $M_\varphi : \mathcal{H}^w \rightarrow \mathcal{H}^w$ defined by $M_\varphi f = \varphi f$ and which, by a standard application of the closed graph theorem (see Lemma 2.5 below), is a bounded operator. Further, (see Lemma 2.6 below), φ is representable by a Dirichlet series in Ω_{σ_0} . Let \mathfrak{M} denote the algebra of multipliers of \mathcal{H}^w viewed as subalgebra of the algebra of bounded operators on \mathcal{H}^w .

In [3], Hedenmalm, Lindqvist and Seip (HLS) classified the multipliers on \mathcal{H}^w in the case that $w_n \equiv 1$ by showing that the multipliers on \mathcal{H}^w were precisely the bounded, analytic functions representable by a Dirichlet series in some right half-plane. More precisely,

Theorem 1.1 ([3]). *If $w_n \equiv 1$ for each n , then*

$$(4) \quad \mathfrak{M} \equiv H^\infty(\Omega_0) \cap \mathcal{D},$$

where $H^\infty(\Omega_0)$ is the space of bounded, analytic functions on Ω_0 and \mathcal{D} is the space of functions representable by a Dirichlet series in some right half-plane.

In [4], McCarthy extended the results of HLS to weights given by a measure.

Theorem 1.2 ([4]). *If μ is a positive Radon measure with $\{0\}$ in its support and if w_n is defined by*

$$(5) \quad w_n = \int_0^\infty n^{-2\sigma} d\mu(\sigma),$$

where n_0 is the smallest natural number for which this integral is finite, then

$$(6) \quad \mathfrak{M} \equiv H^\infty(\Omega_0) \cap \mathcal{D}.$$

In both of the above theorems, $\mathfrak{M} \equiv H^\infty(\Omega_0) \cap \mathcal{D}$ means \mathfrak{M} is isometrically isomorphic to $H^\infty(\Omega_0) \cap \mathcal{D}$.

In this article, estimates on the norms of multipliers on a class of weighted Hilbert spaces of Dirichlet series are established in the cases that the weights satisfy some basic conditions. The weights here are complementary to those generated by measures as considered in [4], overlapping only in the most trivial cases. Examples of weights meeting these conditions include the reciprocals of the divisor function, the sum of the divisors function and Euler's totient function. Indeed, in each of these cases the norm of a multiplier is identified as the supremum norm of the symbol φ over a right half plane.

The Cauchy-Schwarz inequality tells us that norm convergence in \mathcal{H}^w implies pointwise convergence in Ω_{σ_0} and thus, each point $u \in \Omega_{\sigma_0}$ determines a bounded point evaluation on \mathcal{H}^w . The Riesz representation theorem then guarantees the existence of some function k_u in \mathcal{H}^w such that

$$f(u) = \langle f, k_u \rangle_w$$

for each $f \in \mathcal{H}^w$. The function $k : \Omega_{\sigma_0} \times \Omega_{\sigma_0} \longrightarrow \mathbb{C}$ defined by $k(u, z) = k_z(u)$ is called the kernel of \mathcal{H}^w . Since the set $\{w_n^{1/2} n^{-s} : n = 1, 2, \dots\}$ forms an orthonormal basis for \mathcal{H}^w , we have

$$k_z(u) = \langle k_z, k_u \rangle_w = \sum_{n=1}^{\infty} \langle k_z, w_n^{1/2} n^{-s} \rangle_w \langle w_n^{1/2} n^{-s}, k_u \rangle_w.$$

Working out this right-most sum yields

$$(7) \quad k(u, z) = \overline{k_u(z)} = \sum_{n=1}^{\infty} w_n^{-1} n^{-u-\bar{z}}.$$

A sequence of weights $\{w_n\}_{n=1}^{\infty}$ is *multiplicative* if $w_{mn} = w_m w_n$ for each relatively prime pair of natural numbers m and n . As a special case, $\{w_n\}_{n=1}^{\infty}$ is *completely multiplicative* if $w_{mn} = w_m w_n$

for every pair of natural numbers m and n – coprime or not. It is straightforward to see that a multiplicative sequence is determined by its values on the powers of the primes and that a completely multiplicative sequence is determined by its values on 1 and the primes. Note also that $w_1 = w_1 w_1$, so that $w_1 = 0$ or $w_1 = 1$. In the former case, the sequence is identically 0 and nothing interesting happens. Accordingly, we will relegate our discussion to the latter case.

Number theory abounds with examples of multiplicative functions. For example, it is easy to see that the divisor function $d(n)$, which gives the numbers of positive divisors of the natural number n , is multiplicative. Indeed, for each prime p and each natural number k , we have $d(p^k) = k + 1$. If $m = p_1^{\alpha_1} \cdots p_M^{\alpha_M}$, then it is clear that

$$d(m) = d(p_1^{\alpha_1} \cdots p_M^{\alpha_M}) = \prod_{i=1}^M (\alpha_i + 1).$$

If $n = q_1^{\beta_1} \cdots q_n^{\beta_n}$ and if there are no primes dividing both m and n , then

$$d(mn) = d(p_1^{\alpha_1} \cdots p_i^{\alpha_i} q_1^{\beta_1} \cdots q_n^{\beta_n}) = \prod_{i=1}^M (\alpha_i + 1) \prod_{j=1}^N (\beta_j + 1) = d(m)d(n).$$

Let P_N denote the set consisting of the first N primes and denote by $\langle P_N \rangle$ the collection of words generated by the primes in P_N . That is,

$$(8) \quad \langle P_N \rangle = \{2^{\alpha_1} 3^{\alpha_2} \cdots p_N^{\alpha_N} : \alpha_j \in \mathbb{N}\}.$$

Let \mathcal{H}_N^w denote the closure of the subspace of \mathcal{H}^w spanned by the vectors n^{-s} for $n \in \langle P_N \rangle$. That is,

$$(9) \quad \mathcal{H}_N^w = \left\{ f(s) = \sum_{n \in \langle P_N \rangle} a_n n^{-s} : f \in \mathcal{H}^w \right\}.$$

Informally, \mathcal{H}_N^w is the Hilbert space of functions obtained by taking the elements of \mathcal{H}^w and “throwing out” the terms not indexed by $\langle P_N \rangle$. Finally, let π_N denote the projection of \mathcal{H}^w onto \mathcal{H}_N^w .

There are several things that can be said about \mathcal{H}_N^w and π_N .

Lemma 1.3. *Suppose φ is a multiplier on \mathcal{H}^w with*

$$\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Then

(i) $\varphi \in \mathcal{H}^w$;

(ii) $\pi_N \varphi$ is a multiplier on \mathcal{H}_N^w with

$$\pi_N \varphi(s) = \sum_{n \in \langle P_N \rangle} a_n n^{-s};$$

(iii) With an abuse of notation, $\pi_N M_\varphi = M_{\pi_N \varphi}|_{\mathcal{H}_N^w}$; and

(iv)

$$(10) \quad \pi_N k^w(u, z) = \sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-u-\bar{z}}$$

is the reproducing kernel for \mathcal{H}_N^w .

The following is the main result of this article. Here, $|\varphi|_{\Omega_\sigma}$ denotes the supremum of φ on Ω_σ .

Theorem 1.4. *Let $\{w_n\}_{n=1}^\infty$ be a sequence of positive numbers, suppose that φ is a multiplier of \mathcal{H}^w and let $0 \leq \Delta \leq \Delta'$ be real numbers. If*

$$\sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma'} < \infty$$

whenever $\sigma' > \Delta'$ and

$$\sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2\sigma} < \infty$$

for each N whenever $\sigma > \Delta$, then

- (1) each $f \in \mathcal{H}^w$ converges absolutely in $\Omega_{\Delta'}$;
- (2) each $f \in \mathcal{H}_N^w$ converges absolutely in Ω_{Δ} ;
- (3) \mathcal{H}^w is a reproducing kernel Hilbert space (with point evaluations being continuous in $\Omega_{\Delta'}$);
- (4) the sequence $\{\pi_N \varphi\}_{N=1}^\infty$ is uniformly bounded in sup norm on Ω_{Δ} by $\|M_\varphi\|$; and
- (5) φ converges in Ω_{Δ} with

$$(11) \quad |\varphi|_{\Omega_{\Delta}} \leq \|M_\varphi\|.$$

In the other direction, if

- (1) $0 \leq \delta$;
- (2) $\{w_n\}_{n=1}^\infty$ is multiplicative; and
- (3) for each prime p and positive integer k , $w_{p^{k-1}} p^{-2\delta} \geq w_{p^k}$,

then

$$(12) \quad \|M_\varphi\| \leq |\varphi|_{\Omega_\delta}.$$

Remark 1.5. We will see in Section 3 that if $\Delta \geq 0$ and if for each $\sigma > \Delta$ there is a $C_\sigma > 0$ such that

$$(13) \quad w_n^{-1} n^{-2\sigma} \leq C_\sigma$$

for each n , then

$$|\varphi|_{\Omega_\Delta} \leq \|M_\varphi\|.$$

This inequality will be a consequence of the first half of Theorem 1.4.

The case where $w_n \equiv 1$ was considered by HLS in [3]. In that case, the multipliers on \mathcal{H}^w are the Dirichlet series which converge in the full right half plane to a bounded analytic function. Further,

$$\|M_\varphi\| = |\varphi|_{\Omega_0},$$

thus isometrically isomorphically identifying the space of multipliers on \mathcal{H}^w as the space of bounded, analytic functions representable by Dirichlet series in Ω_0 .

In the setting of Theorem 1.2, where the weights are given by a measure on $[0, \infty)$ for which 0 is a point of density, the inequality in Theorem 3.5 is satisfied for each $\Delta > 0$. On the other hand, McCarthy's weights are completely multiplicative only in the case that $w_n \equiv 1$, which follows from Jensen's inequality: If $\{w_n\}_{n=1}^\infty$ is completely multiplicative, then, since

$$w_1 = \mu([0, \infty)) = 1,$$

we can apply Jensen's inequality to the convex function x^2 to see that

$$\left(\int_0^\infty n^{-2\sigma} d\mu(\sigma) \right)^2 = (w_n)^2 = w_{n^2} = \int_0^\infty (n^{-2\sigma})^2 d\mu(\sigma).$$

Equality only occurs when either the integrand is constant a.e.- $[\mu]$ or the convex function being applied to the integral is linear. It follows then, in our case, that μ must be a point mass at some point in $[0, \infty)$. This shows that a completely multiplicative sequence can't come from any positive measure at all except in the case of a point mass. In Section 6, it is shown that the multiplicative number theoretic weights considered earlier are not determined by a measure.

In the present context, the inequality $\|M_\varphi\| \geq |\varphi|_{\Omega_{\Delta+1}}$ follows immediately from standard reproducing kernel machinery. The sharper inequality in (11) is thus the content of this half of the theorem and the proof given here shares similarities with the argument found in the proof of McCarthy's result [4]. However, the method used to obtain this sharper inequality employs in a crucial way a slightly more involved approach to the standard kernel-eigenfunction argument. The reverse

inequality (12) is established by reducing it to the case $w_n \equiv 1$ by first establishing a dilation result. This dilation approach can also be used to recover the upper bound on the multiplier norm for McCarthy's weights (see Theorem 1.2) from the $w_n \equiv 1$ case of Theorem 1.1.

In Section 2, we will be exploring some of the tools necessary to obtain our main result. In Section 3, we will obtain the lower bounds for our multipliers. In particular, we will be examining the case when our weights satisfy the inequality from (3.5). In Section 4 we take a more operator theoretic approach to obtaining the upper bounds for our multipliers. We will then take a look at an alternative way to prove a special case of McCarthy's result in Section 5. The article concludes with Section 6, which contains the precise results and the details of the number theoretic examples introduced earlier in this introduction.

2. PRELIMINARIES

In much the same way that the Cauchy integral formula determines the coefficients for the power series expansion of a function in the Hardy space $H^2(\mathbb{D})$ of the unit disc, there is an integral formula that gives us the coefficients of a Dirichlet series:

Theorem 2.1 ([1]). *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely along the vertical strip $\sigma_0 + it$, then for $x > 0$, we have*

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\sigma_0 + it) x^{\sigma_0 + it} dt = \begin{cases} a_n & \text{if } x = n \\ 0 & \text{otherwise} \end{cases}.$$

The proof of this theorem consists of a simple Fourier coefficient argument.

From Theorem 2.1, it then follows that if two Dirichlet series converge absolutely along some vertical strip on which they agree, then they must in fact be the same Dirichlet series (i.e. have the same coefficients).

Theorem 2.2 ([1]). *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at the point s_0 , then f converges uniformly on each compact set in Ω_{s_0} .*

Corollary 2.3. *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at the point s_0 , then f is analytic in Ω_{s_0} .*

Proof. This follows from Theorem 2.2 and from an application of Morera's theorem. ■

Lemma 2.4. *Let $0 \leq \Delta \leq \Delta'$. If $f \in \mathcal{H}^w$ and*

$$\sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma} < \infty,$$

whenever $\sigma > \Delta'$, then f is analytic in $\Omega_{\Delta'}$. Similarly, if $f \in \mathcal{H}_N^w$ and

$$\sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2\sigma} < \infty,$$

whenever $\sigma > \Delta$, then f is analytic in Ω_{Δ} .

Proof. This is an immediate consequence of Corollary 2.3 and the fact that each $f \in \mathcal{H}^w$ (resp. $f \in \mathcal{H}_N^w$) converges in $\Omega_{\Delta'}$ (resp. Ω_{Δ}) by the Cauchy-Schwarz inequality. ■

The details of the above application of the Cauchy-Schwarz inequality are virtually identical to a later application, so we leave them until a more appropriate time.

Lemma 2.5. *If φ is a multiplier on \mathcal{H}^w , then M_{φ} is continuous (bounded).*

Proof. It suffices to verify the hypotheses of the closed graph theorem. Accordingly, suppose that f_n, g and h are in \mathcal{H}^w , that $f_n \rightarrow g$ and that $M_{\varphi} f_n \rightarrow h$, both in \mathcal{H}^w . Note that every element of \mathcal{H}^w converges absolutely along the vertical strip $\sigma + it$ whenever $\sigma > \sigma_0$. We wish to show that $M_{\varphi} g = h$. Choose z with $\Re(z) > \sigma_0$. Since norm convergence implies point-wise convergence in Ω_{σ_0} (by Cauchy-Schwarz), we have

$$f_n(z) \rightarrow g(z)$$

and

$$(M_{\varphi} f_n)(z) \rightarrow h(z).$$

Now,

$$\begin{aligned} |h(z) - (M_{\varphi} g)(z)| &\leq |h(z) - (M_{\varphi} f_n)(z)| + |(M_{\varphi} f_n)(z) - (M_{\varphi} g)(z)| \\ &= |h(z) - (M_{\varphi} f_n)(z)| + |\varphi(z) f_n(z) - \varphi(z) g(z)| \\ &= |h(z) - (M_{\varphi} f_n)(z)| + |\varphi(z) (f_n(z) - g(z))|. \end{aligned}$$

Letting $n \rightarrow \infty$ shows that $(M_{\varphi} g)(z) = h(z)$. By Theorem 2.1 we see that $M_{\varphi} g = h$, which is what was to be shown. ■

Lemma 2.6. *If φ is a multiplier on \mathcal{H}^w , then $\varphi \in \mathcal{H}^w$.*

This lemma is immediate in the present setting ($n_0 = 1$). It is also true in the more general case that $n_0 > 1$ – as might occur when dealing with weights like McCarthy’s – but the proof is slightly more involved. For simplicity, this case will be avoided.

We close this section with the following remarkable theorem of Schnee, which which will play an important role in the proof of the Theorem 1.4.

Theorem 2.7 ([5]). *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$*

- (1) *converges in some (possibly remote) half-plane Ω_{σ_0} ;*
- (2) *has an analytic continuation to Ω_0 ; and*
- (3) *for each $\epsilon > 0$, satisfies the growth condition*

$$|f(s)| = O(|s|^\epsilon)$$

as $|s| \rightarrow \infty$ in every right half-plane contained in Ω_0 ,

then $\sum_{n=1}^{\infty} a_n n^{-s}$ in fact converges on all of Ω_0 .

3. THE LOWER BOUND

In this section, we will produce a lower bound for M_φ in the case that the weights satisfy the convergence conditions from (15) and (16). As a consequence, we will see that if $\Delta \geq 0$ and if for each $\sigma > \Delta$ there is a $C_\sigma > 0$ such that

$$(14) \quad w_n^{-1} n^{-2\sigma} \leq C_\sigma$$

for each n , then

$$|\varphi|_{\Omega_\Delta} \leq \|M_\varphi\|.$$

We now proceed to the main theorem of this section.

Theorem 3.1. *Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, suppose that φ is a multiplier of \mathcal{H}^w and let $0 \leq \Delta \leq \Delta'$ be real numbers. If*

$$(15) \quad \sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma'} < \infty$$

whenever $\sigma' > \Delta'$ and

$$(16) \quad \sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2\sigma} < \infty$$

for each N whenever $\sigma > \Delta$, then

- (i) *each $f \in \mathcal{H}^w$ converges absolutely in $\Omega_{\Delta'}$;*
- (ii) *each $f \in \mathcal{H}_N^w$ converges absolutely in Ω_Δ ;*
- (iii) *\mathcal{H}^w is a reproducing kernel Hilbert space (with point evaluations being continuous in $\Omega_{\Delta'}$);*
- (iv) *the sequence $\{\pi_N \varphi\}_{N=1}^{\infty}$ is uniformly bounded in sup norm on Ω_Δ by $\|M_\varphi\|$; and*
- (v) *φ converges in Ω_Δ with*

$$|\varphi|_{\Omega_\Delta} \leq \|M_\varphi\|.$$

Proof. An application of the Cauchy-Schwarz inequality shows that point evaluations are continuous in $\Omega_{\Delta'}$: If $s = \sigma + it \in \Omega_{\Delta'}$ and if $f(s) = \sum_{n=1}^{\infty} f_n n^{-s}$, then

$$\begin{aligned} |f(s)| &\leq \sum_{n=1}^{\infty} |f_n| n^{-\sigma} \\ &= \sum_{n=1}^{\infty} |f_n| w_n^{1/2} w_n^{-1/2} n^{-\sigma} \\ &\leq \sqrt{\sum_{n=1}^{\infty} |f_n|^2 w_n} \sqrt{\sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma}} \\ &= \|f\|_{\mathbf{w}} \sqrt{\sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma}}. \end{aligned}$$

Thus, $\mathcal{H}^{\mathbf{w}}$ is a RKHS in which each element is representable by an absolutely convergent Dirichlet series in $\Omega_{\Delta'}$.

Turning our attention to the projected subspace

$$\mathcal{H}_N^{\mathbf{w}} = \left\{ f(s) = \sum_{n \in \langle P_N \rangle} a_n n^{-s} : f \in \mathcal{H}^{\mathbf{w}} \right\},$$

an application of the Cauchy-Schwarz inequality to the function

$$f(s) = \sum_{n \in \langle P_N \rangle} f_n n^{-s}$$

shows – as above – that point evaluations are continuous in Ω_{Δ} . This proves (i), (ii) and (iii). Moreover, since – as noted in Equation (10) – if $k(u, z)$ is the reproducing kernel for $\mathcal{H}^{\mathbf{w}}$, then $\pi_N k(u, z)$ is the reproducing kernel for the projected space $\mathcal{H}^{\mathbf{w}}$ and Lemma 2.4 tells us that each $\pi_N k_u$ is analytic in Ω_{Δ} .

From the standard kernel/eigenfunction argument, we have

$$(17) \quad |\pi_N \varphi|_{\Omega_{\Delta}} \leq \|\pi_N M_{\varphi}|_{\mathcal{H}_N^{\mathbf{w}}}\| \leq \|\pi_N M_{\varphi}\| \leq \|M_{\varphi}\|,$$

with $\pi_N \varphi$ converging in Ω_{Δ} by item (ii). By a normal families argument, the sequence $\{\pi_N \varphi\}_{n=1}^{\infty}$ has a subsequence $\{\pi_{N_j} \varphi\}_{j=1}^{\infty}$ which converges uniformly on compact sets in Ω_{Δ} to some function ψ , analytic in Ω_{Δ} . Hence

$$|\psi|_{\Omega_{\Delta}} \leq \|M_{\varphi}\|.$$

On the other hand, the sequence $\{\pi_{N_j}\varphi\}_{j=1}^\infty$ converges to φ uniformly on compact subsets of $\Omega_{\Delta'}$ by the Cauchy-Schwarz inequality since, if $\varphi(s) = \sum_{n=1}^\infty a_n n^{-s}$ and if $\sigma = \Re(s) > \Delta'$, then

$$|\varphi(s) - \pi_{N_j}\varphi(s)| \leq \sqrt{\sum_{n \in \mathbb{N} \setminus \langle P_{N_j} \rangle} |a_n|^2 w_n} \sqrt{\sum_{n \in \mathbb{N} \setminus \langle P_{N_j} \rangle} w_n^{-1} n^{-2\sigma}},$$

with the right-hand side tending to 0 as $j \rightarrow \infty$. It follows that $\psi = \varphi$ on $\Omega_{\Delta'}$. Since ψ is a bounded analytic continuation of φ into Ω_Δ , Schnee's theorem (a slight variation actually) tells us that the Dirichlet series for φ converges on Ω_Δ . Corollary 2.3 tells us that φ is analytic in Ω_Δ , and as $\varphi = \psi$ in Ω'_Δ , we have $\varphi = \psi$ in Ω_Δ . It follows that $|\varphi|_{\Omega_\Delta} \leq \|M_\varphi\|$. ■

Some facts which arose in the above proof, will be collected in the following theorem.

Theorem 3.2. *Let φ be a multiplier on the space \mathcal{H}^w . If the sequence $\{\pi_N \varphi\}_{N=1}^\infty$ is uniformly bounded by (say) B in supnorm in the half-plane Ω_Δ , then*

- (i) *there is some subsequence $\{\pi_{N_j} \varphi\}_{j=1}^\infty$ converging pointwise to φ in Ω_Δ ;*
- (ii) *φ converges in Ω_Δ ; and*
- (iii) *$|\varphi|_{\Omega_\Delta} \leq B$.*

We now move to a more interesting condition on our weights, which will allow us to obtain the convergence conditions given in Theorem 1.4 and which – as we will see – be more useful in producing bounds for our multipliers when our weights are of a more number theoretic variety. But first, a lemma.

Lemma 3.3. *For each positive integer N , the series given by*

$$\sum_{n \in \langle P_N \rangle} n^{-s}$$

converges in Ω_0 .

Proof. We have

$$\sum_{n \in \langle P_N \rangle} n^{-s} = \prod_{p \in P_N} \frac{1}{1 - p^{-s}}$$

far enough to the right. Since $\sum_{n \in \langle P_N \rangle} n^{-s}$ has an analytic continuation to a bounded function in Ω_ϵ for each $\epsilon > 0$, Schnee's theorem, Theorem 2.7, tells us that $\sum_{n \in \langle P_N \rangle} n^{-s}$ in fact converges in all of Ω_ϵ . Since ϵ was arbitrary, it follows that $\sum_{n \in \langle P_N \rangle} n^{-s}$ converges in Ω_0 . ■

Remark 3.4. *Schnee is not actually needed in this proof, but as it needed in the proof of Theorem 3.1, it seems sensible to use it here as well.*

Theorem 3.5. *Let $\Delta \geq 0$ and a sequence $\{w_n\}_{n=1}^\infty$ of positive numbers be given. If for each $\sigma > \Delta$, there exists a $C_\sigma > 0$ such that*

$$w_n^{-1} n^{-2\sigma} \leq C_\sigma$$

for each n , and if φ is a multiplier of \mathcal{H}^w , then the inequalities in Theorem 3.1 are satisfied by $\Delta + \frac{1}{2}$ and Δ respectively and

$$|\varphi|_{\Omega_\Delta} \leq \|M_\varphi\|.$$

Proof. Let $\epsilon > 0$ and let $\sigma > \Delta + \frac{1}{2} + \epsilon$. The Cauchy-Schwarz inequality gives

$$\begin{aligned} \sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma} &\leq \sum_{n=1}^{\infty} w_n^{-1} n^{-2(\Delta+\epsilon/2)} n^{-1-\epsilon} \\ &\leq C_{\Delta+\epsilon/2} \sum_{n=1}^{\infty} n^{-1-\epsilon} \\ &< \infty, \end{aligned}$$

and we see that the inequality in (15) of Theorem 3.1 is satisfied by $\Delta' = \Delta + \frac{1}{2}$.

We will now show that condition (16) of Theorem 3.1 is satisfied by Δ . To do this, we will make use of a truncated version of the famed Euler product

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

given by

$$\sum_{n \in \langle P_N \rangle} n^{-s} = \prod_{p \in P_N} \frac{1}{1 - p^{-s}}.$$

Let $\epsilon > 0$ and let $\sigma > \Delta + \epsilon$. Observe that

$$\sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2\sigma} \leq \sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2(\Delta+\epsilon/2)} n^{-\epsilon} \leq C_{\Delta+\epsilon/2} \sum_{n \in \langle P_N \rangle} n^{-\epsilon}.$$

The right-most sum converges by Lemma 3.3, so that condition (16) of Theorem 3.1 is satisfied by Δ . It now follows from Theorem 3.1 that

$$|\varphi|_{\Omega_\Delta} \leq \|M_\varphi\|.$$

■

4. THE UPPER BOUND

The second half of Theorem 1.4 is established in this section.

Theorem 4.1. *Let $\{w_n\}_{n=1}^\infty$ be a sequence of positive numbers, let φ be a multiplier of \mathcal{H}^w and let $\delta \geq 0$. If*

- (i) $\{w_n\}_{n=1}^\infty$ is multiplicative; and
- (ii) for each prime p and positive integer k , we have $w_{p^k} \leq p^{-2\delta} w_{p^{k-1}}$,

then

$$\|M_\varphi\| \leq |\varphi|_{\Omega_\delta}.$$

By replacing w_n by $w_n n^{-2\delta}$, it suffices to establish the result for $\delta = 0$. In this case, (ii) can be restated as saying that the sequence $\{w_{p^k}\}_{k=0}^\infty$ is decreasing for each prime p . The case where $w_n \equiv 1$ appears in [3, 2, 4] and is used here to establish this more general result.

Proof. Let \mathcal{H}^0 and k^0 denote the space and kernel corresponding to the weights $w_n \equiv 1$. Note that for u and z in Ω_1 , the kernel $k^0(u, z)$ converges absolutely and is given by

$$(18) \quad k^0(u, z) = \sum_{n=1}^\infty n^{-\tau} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-\tau}},$$

where $\tau = u + \bar{z}$. For the general sequence of weights and for u and z in Ω_{σ_0} (with σ_0 as in (2)), the kernel $k^w(u, z)$ converges absolutely, and has its own product representation given by

$$(19) \quad \prod_{p \text{ prime}} \sum_{j=0}^\infty w_{p^j}^{-1} p^{-j\tau}.$$

Let $m = \max\{1, \sigma_0\}$, let $\tau \in \Omega_m$ and let

$$(20) \quad K(u, z) = \frac{k^w(u, z)}{k^0(u, z)},$$

which is well-defined since $k^0(u, z)$ has no zeros $\Omega_1 \times \Omega_1$. The products for these kernels converge giving,

$$\begin{aligned} K(u, z) &= \prod_{p \text{ prime}} (1 - p^{-\tau}) \sum_{j=0}^\infty w_{p^j}^{-1} p^{-j\tau} \\ &= \prod_{p \text{ prime}} \left(1 + \sum_{j=1}^\infty (w_{p^j}^{-1} - w_{p^{j-1}}^{-1}) p^{-j\tau} \right). \end{aligned}$$

Because the weights are assumed to be decreasing by condition (ii) of Theorem 4.1, we have

$$w_{p^j}^{-1} - w_{p^{j-1}}^{-1} \geq 0,$$

and it follows that $K(u, z)$ is positive semidefinite on $\Omega_m \times \Omega_m$.

From the theory of reproducing kernels, the fact that K is positive semidefinite implies the existence an auxiliary Hilbert space \mathcal{H}^Q and a function $Q : \Omega_m \rightarrow \mathcal{H}^Q$ such that

$$K(u, z) = \frac{k^w(u, z)}{k^0(u, z)} = Q(z)^* Q(u).$$

Multiplying through and rewriting, we have

$$(21) \quad \langle k_u^w, k_z^w \rangle_w = \langle k_u^0, k_z^0 \rangle_0 \langle Q(u), Q(z) \rangle_Q = \langle k_u^0 \otimes Q(u), k_z^0 \otimes Q(z) \rangle_{\otimes}.$$

Define an operator V from the set $\mathcal{S} = \{k_u^w : u \in \Omega_m\}$ to the set $\{k_u^0 \otimes Q(z)\}$ by

$$(22) \quad V k_u^w = k_u^0 \otimes Q(u).$$

We can extend this V by linearity to a map – still denoted by V – to span \mathcal{S} . This V , so defined, is an isometry on span \mathcal{S} . Since span \mathcal{S} is dense in \mathcal{H}^w , V thus extends continuously to an isometry, still denoted by V on all of \mathcal{H}^w into $\mathcal{H}^0 \otimes \mathcal{H}^Q$.

For the multiplier φ on \mathcal{H}^w and for $u \in \Omega_m$ it is well known that

$$M_\varphi^* k_u^w = \overline{\varphi(u)} k_u^w.$$

If φ is unbounded in Ω_0 , then there's nothing to do since then $|\varphi|_{\Omega_0} = \infty$. Otherwise, φ is a bounded Dirichlet series on Ω_0 and it follows from Theorem 1.1 that φ is a multiplier of \mathcal{H}^0 and $\|M_\varphi\|_0 \leq |\varphi|_{\Omega_0}$.

Let $M_{\varphi, w}$ denote multiplication by φ in \mathcal{H}^w and let $M_{\varphi, 0}$ denote multiplication by φ in \mathcal{H}^0 . For $k_u^0 \otimes Q(z)$ in $\mathcal{H}^0 \otimes \mathcal{H}^Q$, we have

$$(23) \quad (M_{\varphi, 0}^* \otimes I)(k_u^0 \otimes Q(z)) = \overline{\varphi(u)} k_u^0 \otimes Q(z),$$

and it follows that

$$(24) \quad V M_{\varphi, w}^* = (M_{\varphi, 0}^* \otimes I) V.$$

It is then easy to see that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^w & \xrightarrow{V} & \mathcal{H}^0 \otimes \mathcal{H}^Q \\ M_{\varphi, w}^* \downarrow & & \downarrow M_{\varphi, 0}^* \otimes I \\ \mathcal{H}^w & \xrightarrow{V} & \mathcal{H}^0 \otimes \mathcal{H}^Q. \end{array}$$

In particular, we see that

$$\|M_{\varphi, w}\| \leq \|V^*\| \|M_{\varphi, 0}\| \|V\| \leq \|M_{\varphi, 0}\| \leq |\varphi|_{\Omega_0},$$

which is what was to be shown. ■

4.1. The Case When $\delta = \Delta$. Now suppose that w_n defines a sequence of positive numbers satisfying the conditions of Theorems 3.1 and 4.1 with $0 \leq \delta \leq \Delta \leq \Delta'$, restated here for convenience:

- (1) $\sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma'} < \infty$ whenever $\sigma' > \Delta'$;
- (2) $\sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2\sigma} < \infty$ for each N whenever $\sigma > \Delta$;
- (3) $\{w_n\}_{n=1}^{\infty}$ is multiplicative; and
- (4) for each prime p , we have $w_{p^k} \leq p^{-2\delta} w_{p^{k-1}}$.

We've seen that $|\varphi|_{\Omega_{\Delta}} \leq \|M_{\varphi}\|$ and $|\varphi|_{\Omega_{\delta}} \geq \|M_{\varphi}\|$. In particular, if $\delta = \Delta$, then the map $M_{\varphi} \mapsto \varphi$ is an isometry from \mathfrak{M} , the space of multipliers on \mathcal{H}^w with operator norm, into $H^{\infty}(\Omega_{\delta}) \cap \mathcal{D}$, the space of functions, bounded and holomorphic in Ω_{δ} which are representable by Dirichlet series in some right half-plane – this space being equipped with the sup norm. The content of this next theorem is to show that this map is in fact surjective – that each $\varphi \in H^{\infty}(\Omega_{\delta}) \cap \mathcal{D}$ gives rise to a multiplier on \mathcal{H}^w .

Theorem 4.2. *If $\varphi \in H^{\infty}(\Omega_{\delta}) \cap \mathcal{D}$, then φ is a multiplier on \mathcal{H}^w .*

Proof. There is no loss in assuming that $\delta = 0$, as before. Since $\varphi \in H^{\infty}(\Omega_0) \cap \mathcal{D}$, Theorem 1.1, tells us that φ is a multiplier on \mathcal{H}^0 . As before, denote the corresponding multiplication operator on \mathcal{H}^0 by $M_{\varphi,0}$. Note that $\mathcal{H}^0 \subseteq \mathcal{H}^w$ since $w_n \leq 1$ for each n – which follows from the facts that our weights are multiplicative and $w_1 = 1$. Now, if $f \in \mathcal{H}^w$ is a Dirichlet series, then with V defined as in the proof of Theorem 4.1, we have

$$\begin{aligned}
 \langle V^*(M_{\varphi,0} \otimes I)Vf, k_{\lambda}^w \rangle_w &= \langle f, V^*(M_{\varphi,0}^* \otimes I)Vk_{\lambda}^w \rangle_w \\
 &= \langle f, V^*(\overline{\varphi(\lambda)}k_{\lambda}^0 \otimes Q(\lambda)) \rangle_w \\
 &= \varphi(\lambda) \langle f, V^*(k_{\lambda}^0 \otimes Q(\lambda)) \rangle_w \\
 &= \varphi(\lambda) \langle f, V^*Vk_{\lambda}^w \rangle_w \\
 &= \varphi(\lambda) \langle Vf, Vk_{\lambda}^w \rangle_w \\
 &= \varphi(\lambda) \langle f, k_{\lambda}^w \rangle_w \\
 &= \langle \varphi f, k_{\lambda}^w \rangle_w.
 \end{aligned}$$

Thus,

$$(25) \quad \langle V^*(M_{\varphi,0} \otimes I)Vf, k_{\lambda}^w \rangle_w = \langle \varphi f, k_{\lambda}^w \rangle_w$$

for each $f \in \mathcal{H}^w$. As the span of the kernel functions k_λ is dense in \mathcal{H}^w , we see that

$$\langle V^*(M_{\varphi,0} \otimes I)Vf, g \rangle_w = \langle \varphi f, g \rangle_w$$

for each $g \in \mathcal{H}^w$ and it follows that $V^*(M_{\varphi,0} \otimes I)V$ is given by multiplication by φ . ■

Corollary 4.3. *If $0 \leq \delta = \Delta \leq \Delta'$ and*

- (1) $\sum_{n=1}^{\infty} w_n^{-1} n^{-2\sigma'} < \infty$ whenever $\sigma' > \Delta'$;
- (2) $\sum_{n \in \langle P_N \rangle} w_n^{-1} n^{-2\sigma} < \infty$ for each N whenever $\sigma > \Delta$;
- (3) $\{w_n\}_{n=1}^{\infty}$ is multiplicative; and
- (4) for each prime p , we have $w_{p^k} \leq p^{-2\delta} w_{p^{k-1}}$.

then

$$\mathfrak{M} \equiv H^\infty(\Omega_\delta) \cap \mathcal{D}.$$

Proof. The map $M_\varphi \mapsto \varphi$ is an isometry by Theorems 3.1 and 4.1 and is surjective by Theorem 4.2. ■

Remark 4.4. *Note that the first two conditions in Corollary 4.3 are satisfied by $\Delta + \frac{1}{2}$ and Δ when the growth condition in Theorem 3.5 is met. Corollary 4.3 may then be restated as follows: If $0 \leq \delta = \Delta$ and*

- (1) *if for each $\sigma > \Delta$ there is a $C_\sigma > 0$ such that*

$$w_n^{-1} n^{-2\sigma} \leq C_\sigma$$

for each n ;

- (2) $\{w_n\}_{n=1}^{\infty}$ is multiplicative; and
- (3) *for each prime p , we have $w_{p^k} \leq p^{-2\delta} w_{p^{k-1}}$,*

then

$$\mathfrak{M} \equiv H^\infty(\Omega_\delta) \cap \mathcal{D}.$$

5. WEIGHTS GIVEN BY MEASURES

Given μ , a Borel probability measure on $[0, \infty)$ with 0 in its support, define

$$(26) \quad w_n = \int_0^\infty n^{-2\sigma} d\mu(\sigma).$$

For ease of exposition, we have restricted our attention the class of probability measures – and corresponding weight sequences – which is less general than that considered by McCarthy in [4].

Theorem 5.1 (McCarthy [4]). *The multiplier algebra of \mathcal{H}^w is isometrically isomorphic to $H^\infty(\Omega_0) \cap \mathcal{D}$, where the norm on $H^\infty(\Omega_0) \cap \mathcal{D}$ is the supremum of the absolute value on Ω_0 .*

Proof. The hypothesis that 0 is the left hand endpoint of the support of μ implies that for every $\sigma > 0$ there is a $C_\sigma > 0$ such that

$$w_n^{-1} n^{-2\sigma} \leq C_\sigma.$$

In particular, if φ is a multiplier of \mathcal{H}^w , then by using Theorem 3.5 in conjunction with (v) of Theorem 3.1, we see that the Dirichlet series for φ converges on Ω_0 and that

$$|\varphi|_{\Omega_0} \leq \|M_\varphi\|.$$

Given $\delta \geq 0$, let \mathcal{H}^δ denote the Hilbert space of Dirichlet series corresponding to the weight sequence $\{n^{-2\delta}\}_{n=1}^\infty$, let $\|f\|_\delta$ and $\langle f, g \rangle_\delta$ denote the norm and inner product in \mathcal{H}^δ respectively and let \mathcal{F} denote the set of all continuous, bounded functions from $[0, \infty)$ into \mathcal{H}^0 . Observe that \mathcal{H}^0 includes (contractively) into \mathcal{H}^δ as well as into \mathcal{H}^w . In particular, for $F \in \mathcal{F}$,

$$\int_0^\infty \|F(\delta)\|_\delta^2 d\mu(\delta) \leq \int_0^\infty \|F(\delta)\|_0^2 d\mu(\delta) \leq C^2,$$

where C is a bound for F in the sense that $\|F(\delta)\| \leq C$ for each $\delta \geq 0$.

On \mathcal{F} , consider the inner product given by

$$(27) \quad \langle F, G \rangle = \int_0^\infty \langle F(\delta), G(\delta) \rangle_\delta d\mu(\delta)$$

and let \mathcal{F}^2 denote the resulting Hilbert space. Define W on the dense subset \mathcal{H}^0 of \mathcal{H}^w by inclusion as constant functions:

$$(28) \quad (Wf)(\delta) = f.$$

Observe that, for finite Dirichlet series $f(s) = \sum_{n=1}^m f_n n^{-s}$ and $g = \sum_{n=1}^m g_n n^{-s}$ in \mathcal{H}^0 , we have

$$\begin{aligned} \langle Wf, Wg \rangle &= \int_0^\infty \langle f, g \rangle_\delta d\mu(\delta) \\ &= \sum_{n=1}^m f_n \overline{g_n} \int_0^\infty n^{-2\delta} d\mu(\delta) \\ &= \sum_{n=1}^m f_n \overline{g_n} w_n. \\ &= \langle f, g \rangle_w. \end{aligned}$$

It follows that W extends to an isometry (still denoted by W) from \mathcal{H}^w into \mathcal{F}^2 .

As already noted, the Dirichlet series for φ converges on all of Ω_0 . It follows that if f is a finite Dirichlet series (finitely many nonzero terms), then $\varphi f \in \mathcal{H}^0$. Hence,

$$(W\varphi f)(\delta) = \varphi f.$$

On the other hand, given $F \in \mathcal{F}$, the function $JF(\delta) = \varphi F(\delta)$ is in \mathcal{F} since φ is a multiplier of \mathcal{H}^0 . The HLS result with δ in place of 0 gives,

$$\begin{aligned} \|JF\|^2 &= \int_0^\infty \|\varphi F(\delta)\|_\delta^2 d\mu(\delta) \\ &\leq \int_0^\infty |\varphi|_{\Omega_\delta}^2 \|F(\delta)\|_\delta^2 d\mu(\delta) \\ &\leq |\varphi|_{\Omega_0}^2 \int_0^\infty \|F(\delta)\|_\delta^2 d\mu(\delta) \\ &= |\varphi|_{\Omega_0}^2 \|F\|^2. \end{aligned}$$

Thus J extends to a bounded operator on \mathcal{F}^2 with $\|J\| \leq |\varphi|_{\Omega_0}$. To complete the proof, observe that

$$WM_\varphi = JW,$$

from which we can see that

$$\|M_\varphi\| \leq \|J\|.$$

■

6. EXAMPLES

We close this article by looking at some examples of weight sequences defined by certain well-known arithmetic functions. We've already seen that – with the exception of weights coming from point-masses – completely multiplicative sequences can't arise from a measure like McCarthy's. The Jensen's inequality argument used in the introduction doesn't carry over to the case when the weights are multiplicative, but not completely so.

Definition 6.1. *The divisor function – denoted by $d(n)$ – gives the number of positive divisors of the natural number n . If $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, then as we saw in the introduction,*

$$d(n) = \prod_{j=1}^m (\alpha_j + 1).$$

The sum of divisors function – denoted by $\sigma(n)$ – gives the sum of the divisors of the natural number n . If $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, then

$$\sigma(n) = \prod_{j=1}^m \left(\frac{p_j^{\alpha_j+1} - 1}{p_j - 1} \right).$$

The Euler totient function – denoted by $\phi(n)$ – tells how many positive integers less than n are coprime to n . If $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, then

$$\phi(n) = n \prod_{j=1}^m \left(1 - \frac{1}{p_j}\right).$$

Each of these functions is multiplicative; none are completely so. In this section, we will be working with sequences given by the reciprocals of these functions. Any such sequence is automatically multiplicative.

Theorem 6.2. *If w_n is given by the reciprocal of $d(n)$, $\sigma(n)$ or $\phi(n)$, then there is no positive measure μ such that*

$$(29) \quad w_n = \int_0^\infty n^{-2\sigma} d\mu(\sigma).$$

On the other hand, in each case there is a δ such that \mathfrak{M} is isometrically isomorphic to $H^\infty(\Omega_\delta) \cap \mathcal{D}$.

Before moving into the proof of this theorem, some preliminaries are in order.

Lemma 6.3. *For each natural number n ,*

$$\frac{6}{\pi^2} < \frac{\sigma(n)\phi(n)}{n^2} < 1.$$

Growth Rates: The following growth rates, the first of which is known as Grönwall's Theorem, hold.

1. Grönwall's Theorem:

$$(30) \quad \limsup_n \frac{\sigma(n)}{n \ln \ln n} = e^\gamma$$

where γ is the Euler-Mascheroni constant.

2. For each $\epsilon > 0$,

$$(31) \quad \sigma(n) = O(n^{1+\epsilon}).$$

3.

$$(32) \quad \limsup_n \frac{\phi(n)}{n} = 1;$$

4.

$$(33) \quad \liminf_n \frac{\phi(n)}{n} = 0.$$

We are now in a position to prove Theorem 6.2.

Proof. Suppose first that $w_n = \frac{1}{d(n)}$. Assume by way of contradiction that there is some positive measure μ such that

$$w_n = \frac{1}{d(n)} = \int_0^\infty n^{-2\sigma} d\mu(\sigma).$$

Let $\{p_1 < p_2 < p_3 < \cdots\}$ be the primes for which this integral is finite. Then

$$\frac{1}{2} = \frac{1}{d(p_j)} = \int_0^\infty p_j^{-2\sigma} d\mu(\sigma).$$

Letting $j \rightarrow \infty$ and using the dominated convergence theorem, we see that $\frac{1}{2} = \mu(\{0\})$. Now, apply the same argument with the sequence $\{p_1^2 < p_2^2 < p_3^2 < \cdots\}$, along with the fact that $\frac{1}{d(p_j^2)} = \frac{1}{3}$ to get $\mu(\{0\}) = \frac{1}{3}$ – a contradiction.

Suppose now that $w_n = \frac{1}{\sigma(n)}$ and again, assume by way of contradiction that w_n is given by a measure as in Equation (5). Grönwall's theorem implies that there is some sequence $\{n_j\}_{j=1}^\infty$ such that

$$\lim_{j \rightarrow \infty} \frac{n_j \ln \ln n_j}{\sigma(n_j)} = e^{-\gamma}.$$

Hence

$$\begin{aligned} e^{-\gamma} &= \lim_{j \rightarrow \infty} \frac{n_j \ln \ln n_j}{\sigma(n_j)} \\ &= \lim_{j \rightarrow \infty} \int_0^\infty n_j^{1-2\sigma} \ln \ln n_j d\mu(\sigma) \\ &\geq \lim_{j \rightarrow \infty} \int_{[0, 1/2]} n_j^{1-2\sigma} \ln \ln n_j d\mu(\sigma) \\ &\geq \lim_{j \rightarrow \infty} \mu([0, 1/2]) \ln \ln n_j. \end{aligned}$$

It follows that $\mu([0, 1/2]) = 0$.

Since

$$\frac{6}{\pi^2} < \frac{\sigma(n)\phi(n)}{n^2} < 1$$

(by Lemma 6.3), (32) implies the existence of a sequence $\{n_i\}_{i=1}^\infty$ such that

$$\frac{6}{\pi^2} \leq \frac{\sigma(n_i)}{n_i} \leq 2,$$

for each i , whence

$$2 \leq \limsup_i \frac{n_i}{\sigma(n_i)}.$$

Thus

$$\begin{aligned}
2 &\leq \limsup_i \frac{n_i}{\sigma(n_i)} \\
&= \limsup_i \int_0^\infty n_i^{1-2\sigma} d\mu(\sigma) \\
&= \limsup_i \left(\int_{[0,1/2]} n_i^{1-2\sigma} d\mu(\sigma) + \int_{(1/2,\infty)} n_i^{1-2\sigma} d\mu(\sigma) \right) \\
&= \limsup_i \left(\int_{(1/2,\infty)} n_i^{1-2\sigma} d\mu(\sigma) \right) \\
&= 0,
\end{aligned}$$

with the second-to-last step coming from the fact that $\mu([0, 1/2]) = 0$ and the last step coming from Fatou's lemma. This contradiction proves that w_n so defined can't come from a measure.

Finally, let's $w_n = \frac{1}{\phi(n)}$ and again, suppose by way of contradiction, that there is a μ satisfying Equation (5). By (32), there is a sequence $\{n_i\}_{i=1}^\infty$ such that

$$\lim_{i \rightarrow \infty} \frac{n_i}{\phi(n_i)} = 1.$$

It follows that

$$\begin{aligned}
1 &= \lim_{i \rightarrow \infty} \frac{n_i}{\phi(n_i)} \\
&= \lim_{i \rightarrow \infty} \int_0^\infty n_i^{1-2\sigma} d\mu(\sigma) \\
&= \lim_{i \rightarrow \infty} \left(\int_{[0,1/2]} n_i^{1-2\sigma} d\mu(\sigma) + \mu(\{1/2\}) + \int_{(1/2,\infty)} n_i^{1-2\sigma} d\mu(\sigma) \right).
\end{aligned}$$

Now, the first integral in the parentheses must be 0 and the second integral in the parentheses must tend to 0 by the dominated convergence theorem, so that $\mu(\{1/2\}) = 1$. However, running through the same argument with a sequence furnished by (33) shows that $\mu(\{1/2\}) = \infty$ – a contradiction.

So, if w_n is given by the reciprocal of any one of the three aforementioned multiplicative functions, then it can't come from any measure as in Equation (5).

We now move to the second claim of Theorem 6.2 – that in each case, there is a δ such that \mathfrak{M} is isometrically isomorphic to $H^\infty(\Omega_\delta) \cap \mathcal{D}$. Let's first examine the case where $w_n = \frac{1}{d(n)}$. In this case, $d(n) = o(n^\epsilon)$ for each positive ϵ (see [6]). So, for each $\sigma > 0$, $w_n^{-1} n^{-2\sigma} \rightarrow 0$ as $n \rightarrow \infty$, and we can certainly find a C_σ such that $w_n^{-1} n^{-2\sigma} \leq C_\sigma$ for all n . The other conditions in Remark 4.4

are satisfied: w_n is clearly multiplicative and $w_{p^k} = \frac{1}{d(p^k)} = \frac{1}{k+1}$ is decreasing with k for each prime p . Therefore,

$$\mathfrak{M} \equiv H^\infty(\Omega_0) \cap \mathcal{D}.$$

If instead $w_n = \frac{1}{\phi(n)}$, then the conditions of Remark 4.4 are satisfied with $\delta = \frac{1}{2}$ as from (32): There is some $K > 0$ such that $\frac{\phi(n)}{n} \leq K$, which implies that

$$\phi(n)n^{-1} = \phi(n)n^{-2(1/2)} < K$$

for each n , w_n is multiplicative and $w_{p^k} \leq p^{-1}w_{p^{k-1}}$. Thus

$$\mathfrak{M} \equiv H^\infty(\Omega_{1/2}) \cap \mathcal{D}.$$

Finally, suppose that $w_n = \frac{1}{\sigma(n)}$. To see that the conditions of Remark 4.4 are satisfied when $\delta = \frac{1}{2}$, note that w_n is multiplicative and $w_{p^k} = \frac{p-1}{p^{k+1}-1} \leq p^{-1} \frac{p-1}{p^k-1}$ for each k and each prime p . Now, for each $\epsilon > 0$, (31) implies the existence of some positive number K_ϵ such that $\frac{\sigma(n)}{n^{1+\epsilon}} \leq K_\epsilon$ for each n , so that

$$\sigma(n)n^{-2(\frac{1}{2}(1+\epsilon))} < K_\epsilon.$$

Hence

$$\mathfrak{M} \equiv H^\infty(\Omega_{1/2}) \cap \mathcal{D}.$$

■

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